MINIMAL SURFACES ASSOCIATED WITH NONPOLYNOMIAL CONTACT SYMMETRIES

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ABSTRACT. Two infinite sequences of minimal surfaces in space are constructed using symmetry analysis. In particular, explicit formulas are obtained for the self-intersecting minimal surface that fills the trefoil knot.

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Introduction. In this paper we consider the Euler–Lagrange minimal surface equation

$$\mathcal{E}_{\min \Sigma} = \left\{ (1 + u_y^2)\, u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)\, u_{yy} = 0 \right\} \tag{1}$$

whose solutions describe two-dimensional minimal surfaces $\Sigma \subset \mathbb{E}^3$ in nonparametric form $\Sigma = \{z = u(x, y)\}$, here x, y, z are the Cartesian coordinates. We construct two infinite sequences of the minimal surfaces related to nonpolynomial contact symmetries of Eq. (1).

Remark 1. Although the graphs of solutions for Eq. (1) determine the minimal surfaces only locally such that the projections of their tangent planes to $0xy$ are nondegenerate, this is not restrictive for our reasonings. The minimal surfaces constructed in section 2 are self-intersecting, being in fact described by multi-valued solutions of Eq. (1) and admitting singular points.

The paper is organized as follows. In section 1 we describe the generators and the commutator relations of the contact symmetry algebra for $\mathcal{E}_{\min \Sigma}$. We provide examples of the contact non-point generators and indicate the recursion operators for the commutative Lie subalgebra of sym $\mathcal{E}_{\min \Sigma}$. In section 2 we show that any surface which is invariant w.r.t. a contact non-point symmetry flow is always a plane, although non-planar minimal surfaces in space are assigned to the same symmetry generators by the inverse Legendre transformation. Thus we construct two sequences of the minimal surfaces associated with nonpolynomial contact symmetries of $\mathcal{E}_{\min \Sigma}$; one of the sequences starts with the helicoid [4] in $\mathbb{E}^3$. The recursions for the symmetries provide

discrete transformations between the surfaces, while the generators themselves determine their continuous transformations. In particular, we obtain explicit formulas for the self-intersecting minimal surface Σ_6 that fills the trefoil knot; this surface succeeds the helicoid Σ_5 with respect to the recursion relations.

Remark 2. The very idea to construct an infinite sequence of minimal surfaces with no restrictions upon the boundary conditions can be easily fulfilled by using the Ennepert–Weierstrass representation [4] that assigns the surfaces to arbitrary complex-analytic functions. The objective of this note is that the geometry of Eq. (1) suggests a *natural* discrete proliferation scheme based on the symmetry approach.

The resulting surfaces contained in Appendix A seem to be relevant in Natural sciences (hydromechanics, bionics, or chemistry); strangely, these particular solutions given in parametric representation are not met in classical textbooks and reviews on the topic [4, 6].

1. CONTACT SYMMETRIES OF THE MINIMAL SURFACE EQUATION

The Ennepert–Weierstrass representation [4] yields that the symmetry group of the minimal surface equation is the conformal group, which is a semi-direct product of the Möbius group and the component that corresponds to complex-analytic functions (also, the full symmetry group incorporates the dilatation). The Möbius subgroup corresponds to the group of rotations of $\mathbb{E}^3$ owing to the isomorphism $\mathfrak{sl}(2) \simeq \mathfrak{so}(3)$. In this section we interpret the above assertion in view of the Legendre transformation that brings Eq. (1) to linear form.

Let us recall that equation (1) is mapped to the linear elliptic equation

$$\mathfrak{L}(\mathcal{E}_{\min\Sigma}) = \left\{ \left(1 + p^2\right) \phi_{pp} + 2pq\, \phi_{pq} + \left(1 + q^2\right) \phi_{qq} = 0 \right\} \tag{2}$$

by the Legendre transformation

$$\mathfrak{L} = \{ \phi = xu_x + yu_y - u,\ p = u_x,\ q = u_y \}.$$

The inverse Legendre transformation $\mathfrak{L}^{-1} = \{ x = \phi_p,\, y = \phi_q,\, u = p\phi_p + q\phi_q - \phi \}$ assigns the minimal surfaces Σ in parametric form to solutions of Eq. (2).

Each symmetry of Eq. (2) corresponds to a symmetry transformation of Eq. (1). Recall that the determining relation $L_\varphi(F) = 0$ on $\mathcal{E} = \{F = 0\}$ for the infinitesimal symmetries φ of any linear differential equation $\mathcal{E}$ coincides with the equation itself, here L_φ is the evolutionary vector field with the generator φ (see [5]). Therefore it is quite natural that the symmetry algebra of Eq. (1) incorporates the set of solutions

$$\varphi(u_x, u_y) = \phi(p, q) \tag{3}$$

of the linear equation (2). Hence follows the description of contact symmetry algebra for the minimal surface equation (1).

Proposition 1. *The Lie algebra* $\operatorname{sym} \mathcal{E}_{\min\Sigma}$ *of contact symmetries of the minimal surface equation* (1) *is generated by solutions* $\varphi(u_x,\ u_y)$ *of Eq.* (2), *in particular, by the shift* $\varphi_1 = 1$ *and the translations* $\varphi_2^i = u_{x^i}$ *along* $x^1 \equiv x$ *and* $x^2 \equiv y$, *by the rotations* $\varphi_3^{12} = yu_x - xu_y$ *and* $\varphi_3^i = x^i + uu_{x^i}$, *here* $i = 1, 2$, *and by the dilatation* $\varphi_4 = u - xu_x - yu_y$.

Example 1. In [3] two infinite sequences of the symmetry generators $\varphi(u_x,\, u_y)$ for the minimal surface equation were constructed. It was postulated that the functions φ are polynomial in u_y; then for each degree $k \geq 0$ of the polynomials there are two solutions. The initial terms of these sequences are

$$\varphi_1 = 1, \quad \varphi_2^1 = u_x, \qquad\qquad \varphi_2^2 = u_y, \quad \varphi_5 = u_y \arctan u_x,$$

$$\varphi_6 = \frac{u_x u_y^2}{1 + u_x^2} + \arctan u_x, \qquad\qquad \varphi_7 = \frac{u_y^2}{1 + u_x^2} - u_x \arctan u_x,$$

$$\varphi_8 = \frac{u_x u_y^3}{(1 + u_x^2)^2} + \frac{3}{2} \cdot \frac{u_x u_y}{1 + u_x^2}, \qquad\qquad \varphi_9 = \frac{u_x^2 - 1}{(1 + u_x^2)^2} \cdot u_y^3 - \frac{3 u_y}{1 + u_x^2}.$$

The generators φ_k depend rationally on u_x for all $k \geq 8$. We conjecture that none of the contact symmetries φ_k is Noether whenever $k \geq 5$.

The identification (3) yields that a sequence of solutions $\phi(p, q)$ is obtained whenever a recursion for the contact non-point symmetries $\varphi(u_x, u_y)$ of Eq. (1) is known. We claim that three local recursion operators for this component of $\operatorname{sym} \mathcal{E}_{\min\Sigma}$ are determined by the adjoint representation of the symmetry algebra itself. Now we study these aspects in more detail.

The commutation relations for the seven point symmetries $\varphi_1, \, \ldots, \, \varphi_4$, see Proposition 1, were derived in [1]. Let us indicate the commutation properties of the contact symmetries that originate from Eq. (2).

Lemma 2. *Assume that $\varphi'(u_x, u_y)$ and $\varphi''(u_x, u_y)$ are the generators of evolutionary vector fields $\mathrm{L}_{\varphi'}$ and $\mathrm{L}_{\varphi''}$. Then their Jacobi bracket $\{\varphi', \, \varphi''\}$ is always trivial.*

Proposition 3. *All contact symmetries $\varphi(u_x, \, u_y) \in \operatorname{sym} \mathcal{E}_{\min\Sigma}$ of the minimal surface equation $\mathcal{E}_{\min\Sigma}$ commute. Also, the following relations hold*:

$$\{\varphi_3^{12}, \varphi\} = u_x \frac{\partial \varphi}{\partial u_y} - u_y \frac{\partial \varphi}{\partial u_x}, \qquad \{\varphi_4, \varphi\} = -\varphi,$$

$$\{\varphi_3^i, \varphi\} = -u_{x^i} \varphi + (1 + u_{x^i}^2) \frac{\partial \varphi}{\partial u_{x^i}} + u_x u_y \frac{\partial \varphi}{\partial u_{x^{3-i}}}.$$

The Lie subalgebra $\mathfrak{h}$ generated by the solutions $\varphi(u_x, \, u_y)$ of Eq. (2) is the radical of the contact symmetry algebra $\operatorname{sym} \mathcal{E}_{\min\Sigma}$ for Eq. (1).

Corollary 4. The mappings $\operatorname{ad}_{\varphi_3^{12}}$ and $\operatorname{ad}_{\varphi_3^i} : \mathfrak{h} \to \mathfrak{h}$ define the local recursion operators on the Lie subalgebra $\mathfrak{h} \subset \operatorname{sym} \mathcal{E}_{\min\Sigma}$.

Remark 3. The symmetries introduced in Example 1 are proliferated according to the following diagram [2]:

$$
\begin{array}{ccccccc}
\varphi_5 & \xrightarrow{\operatorname{ad}_{\varphi_3^2}} & \varphi_6 & \xrightarrow{\operatorname{ad}_{\varphi_3^1}} & \varphi_7 & \xrightarrow{-\frac{1}{2}\operatorname{ad}_{\varphi_3^{12}}} & \varphi_8 - \frac{1}{2}\varphi_5 \\
{\scriptstyle \operatorname{ad}_{\varphi_3^1}} \downarrow & & & & & & \downarrow {\scriptstyle -\operatorname{ad}_{\varphi_3^1}} \\
\varphi_2^2 & \xrightarrow{\operatorname{ad}_{\varphi_3^2}} & \varphi_1 & \xrightarrow{-\operatorname{ad}_{\varphi_3^1}} & \varphi_2^1 & & \varphi_9 + \frac{7}{2}\varphi_2^2.
\end{array}
$$

Two infinite sequences of the contact symmetries are obtained [3] from φ_6 and φ_7 by multiple application of the recursion $\mathrm{ad}_{\varphi_3^{12}}$. From the above diagram it follows that the symmetry φ_5 is the 'seed' generator for both sequences.

2. THE MINIMAL SURFACES ASSOCIATED WITH THE CONTACT SYMMETRIES

Continuing the line of reasonings, we see that any contact symmetry $\varphi(u_x, u_y) \in \mathfrak{h}$ of Eq. (1) determines the minimal surfaces Σ using two different methods:

(1) Recall that $\phi(p, q)$ defined in (3) is a solution of Eq. (2), hence the inverse image $\mathfrak{L}^{-1}(\phi)$ of the graph of ϕ with respect to the Legendre transformation is a minimal surface in parametric representation.
(2) The symmetry reduction $\mathcal{E}_{\min \Sigma} \cap \{\varphi = 0\}$ of Eq. (1) by a generator $\varphi \in \mathfrak{h} \subset \mathrm{sym}\, \mathcal{E}_{\min \Sigma}$ determines the φ-invariant surface in $\mathbb{E}^3$.

Example 2. Consider the solution $\phi_5 = q \arctan p$ of Eq. (2). Using the first method, we obtain the helicoid $\{z = x \tan y\} \subset \mathbb{E}^3$ whose axis is $0y$. The minimal surface which is invariant w.r.t. the symmetry $\varphi_5 = u_y \arctan u_x$ is a plane. This is a particular case of the following general property of the surfaces.

Proposition 5. *Suppose that a minimal surface Σ is invariant w.r.t. a contact symmetry $\varphi(u_x, u_y)$. Then Σ is a plane.*

Proof. Consider the constraint $\varphi = 0$. By the implicit function theorem, we have $u_y = f(u_x)$ almost everywhere. Therefore, $u_{yy} = (f'(u_x))^2 \cdot u_{xx}$ and from Eq. (1) we obtain the equation

$$\left((1 + u_x^2) \cdot (f'(u_x))^2 - 2u_x\, f(u_x) \cdot f'(u_x) + (1 + f^2(u_x)) \right) \cdot u_{xx} = 0.$$

Hence either $u_{xx} = 0$ and we have $u_x = \mathrm{const}$, $u_y = f(u_x) = \mathrm{const}$, or u_x satisfies the algebraic equation whose solutions are $u_x = \mathrm{const}$ and therefore $u_y = f(u_x) = \mathrm{const}$ again. $\qquad \square$

In what follows we focus on the first method for constructing the minimal surfaces, that is, $\Sigma = \mathfrak{L}^{-1}(\phi(p, q))$. Using Proposition 3 and Corollary 4, we obtain two sequences of solutions $\phi_k(p, q)$ of Eq. (2), which are polynomial in q and which are nonpolynomial in p for $k \geq 5$. By Remark 3, both sequences are obtained from the generating section $\phi_5(p, q)$. First let us list the solutions ϕ_{10}, ϕ_{11} and ϕ_{12}, ϕ_{13} of Eq. (2), which are polynomials in q of degrees 4 and 5, respectively. We have

$$\phi_{10} = \frac{p^3 - 3p}{(1 + p^2)^3} \cdot q^4 + \frac{3}{2} \cdot \frac{p^5 - 2p^3 - 3p}{(1 + p^2)^3} \cdot q^2 - \frac{3}{2} \cdot \frac{p}{1 + p^2},$$

$$\phi_{11} = \frac{3p^2 - 1}{(1 + p^2)^3} \cdot q^4 + \frac{3}{2} \cdot \frac{3p^4 + 2p^2 - 1}{(1 + p^2)^3} \cdot q^2 + \frac{3}{2} \cdot \frac{p^2}{1 + p^2},$$

$$\phi_{12} = \frac{p^4 - 6p^2 + 6}{(1 + p^2)^4} \cdot q^5 + \frac{11p^6 - 49p^4 - 51p^2 + 9}{6(1 + p^2)^4} \cdot q^3 + \frac{2p^8 - 3p^6 - 11p^4 - 5p^2 + 1}{2(1 + p^2)^4} \cdot q,$$

$$\phi_{13} = \frac{p^3 - p}{(1 + p^2)^4} \cdot q^5 + \frac{21p^5 + 2p^3 - 19p}{12(1 + p^2)^4} \cdot q^3 + \frac{3p^7 + 4p^5 - p^3 - 2p}{4(1 + p^2)^4} \cdot q.$$

The resulting surfaces can be easily visualized using standard software, e.g., by modifying this sample code for MAPLE:

```
phi6:=p*q^2/(1+p^2)+arctan(p);
x:=diff(phi6,p); y:=diff(phi6,q); z:=p*x+q*y-phi6;
plot3d([x,y,z],p=-2..2,q=-2..2,grid=[50,50]);
```
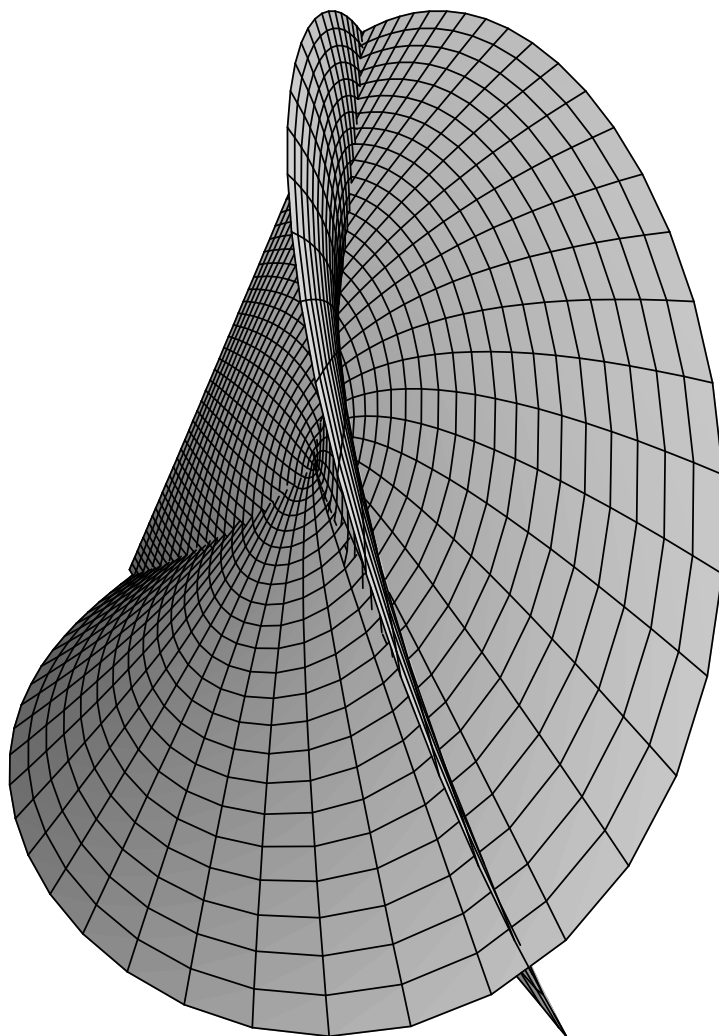
We discover that the boundary of the surface $\Sigma_6 = \mathfrak{L}^{-1}(\phi_6)$ is supported by the trefoil knot such that the self-intersecting surface Σ_6 fills its interiour. The regular minimal surface $\Sigma_7 = \mathfrak{L}^{-1}(\phi_7)$ resembles a flying bird. The surfaces Σ_8 and Σ_{13} are propeller-like. The family Σ_{9-12} provides the shapes of pearl shells; the origin is a singular point for them, and their self-intersections divide the space $\mathbb{E}^3$ in different number of cells for different subscripts k. The corresponding pictures are given in Appendix A.

REFERENCES

[1] N. Bîlă, Lie groups applications to minimal surfaces PDE, Diff. Geometry – Dynam. Systems 1 (1999) no.1, 1–9.

[2] A.V. Kiselev, On a symmetry reduction of the minimal surface equation, Proc. XXVII Conf. of Young Scientists, April 2005. Faculty of Mathematics and Mechanics, Lomonosov MSU, Moscow (2006), 67-71.

[3] A.V. Kiselev, G. Manno, On the symmetry structure of the minimal surface equation, Proc. IX Conf. 'Differential Geometry and Its Applications,' August 2004. Charles Univ., Prague (2005), 483-490. `arXiv:math.DG/0410557`

[4] J.C.C. Nitsche, Vorlesungen über Minimalflächen. Springer, Berlin, 1974.

[5] P.J. Olver, Applications of Lie groups to differential equations, 2nd ed., Springer, Berlin, 1993.

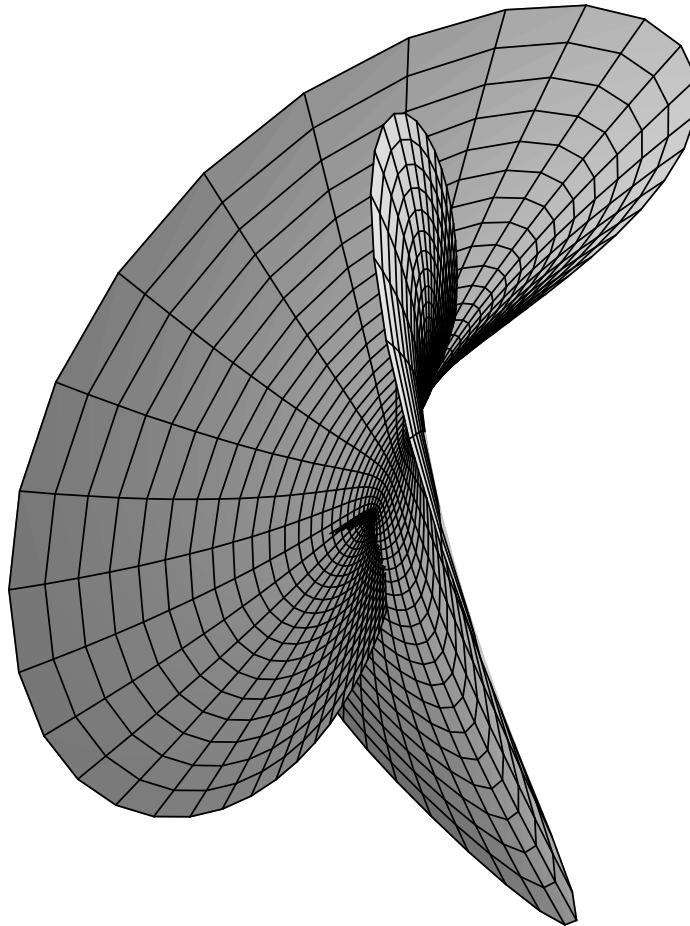[6] Osserman R., A survey of minimal surfaces. Van Nostrand Reinhold, NY, 1969.

Appendix A. The plots of minimal surfaces
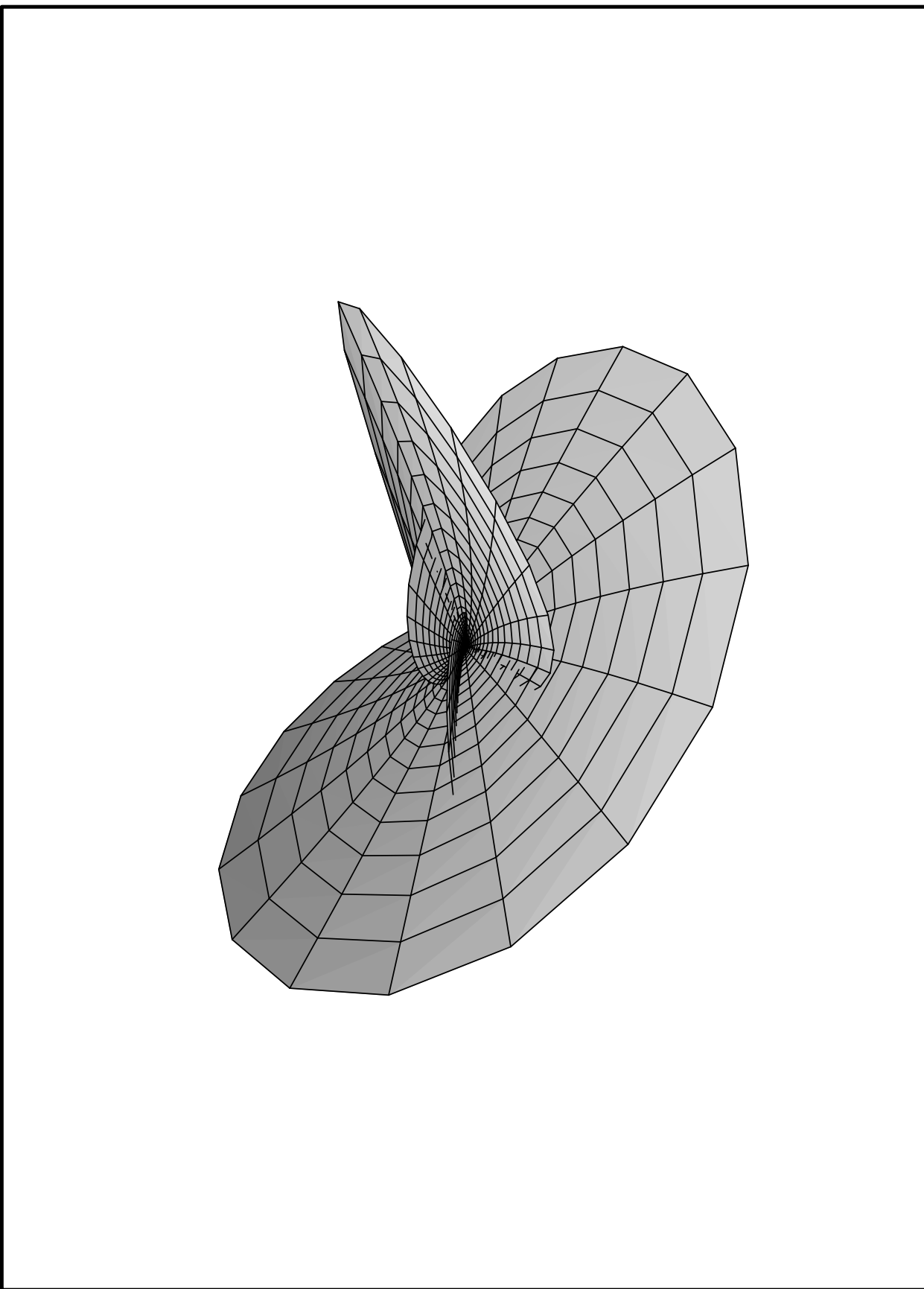
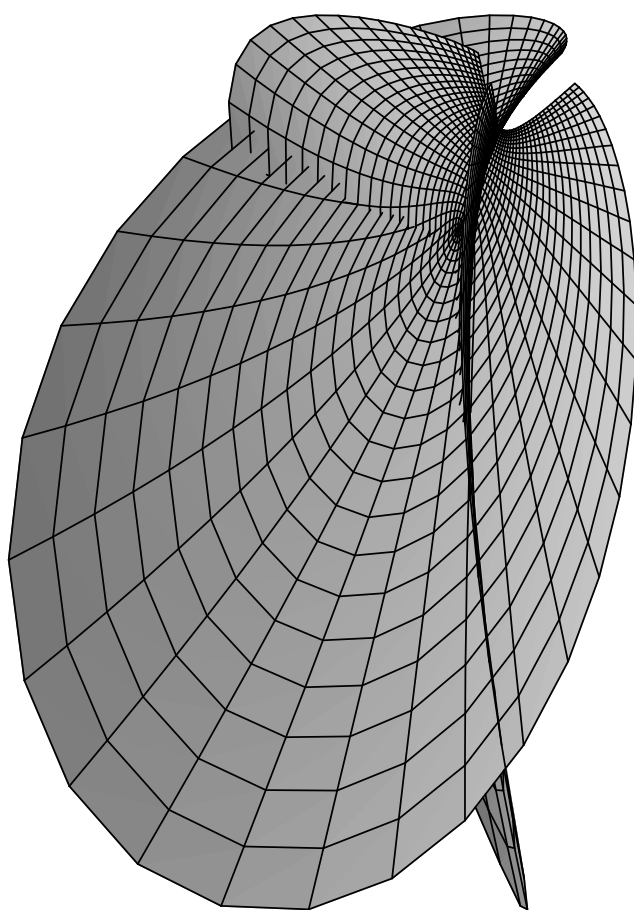A.1. **The surface Σ_6: trefoil knot.**

A.2. **The surface Σ_7: flying bird.**

A.3. **The surfaces Σ_8 and Σ_{13}: propellers.**

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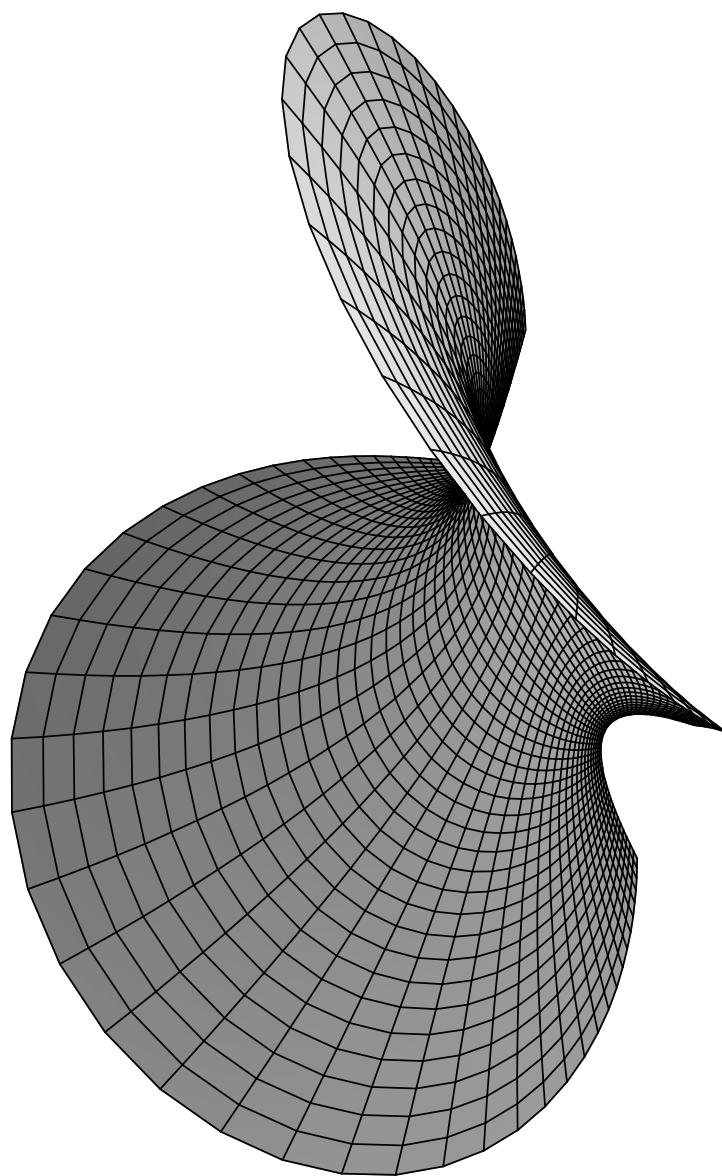
The surface Σ_{13}:

A.4. The surfaces Σ_{9-12}: pearl shells.

The surface Σ_{10}:

The surface Σ_{11}:

The surface Σ_{12}: